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LETTER TO THE EDITOR

Dynamics of the Born–Infeld dyons

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Abstract. The approach to the dynamics of a charged particle in the Born–Infeld nonlinear electrodynamics developed in Chruściński (1998 *Phys. Lett.* A **240** 8) is generalized to include Born–Infeld dyons. Both Hamiltonian and Lagrangian are constructed. Some similarities with the BPS mass formula and topological field theory are discussed.

1. Introduction

Recently, Born–Infeld nonlinear electrodynamics [2] has found a deep application in string theory and p-brane physics (see e.g. recent papers [3] for a review and references). However, in this letter we consider not branes but *ordinary* point particles coupled to the Born–Infeld field (this was actually the original motivation of Born and Infeld [2]).

In [1] it was shown how to consistently describe the classical dynamics of electrically charged particles. Of course this problem is trivial when one considers *test* particles. Then one simply takes a standard interaction Lagrangian '$j^\mu A_\mu$' and the Lorentz equations of motion obviously follow. However, the description of *true* particles leads (as in the Maxwell theory) to basic difficulties. Due to the nonlinearity it is impossible to derive separate equations of motion for a particle corresponding, for example, to the celebrated Lorentz–Dirac equation in the Maxwell theory. The main result of [1] consists of the observation that information about a particle's dynamics is entirely encoded into some boundary condition for field variables which has to be satisfied along a particle's trajectory (in [1] it was called the *dynamical condition*).

Now, since the Born–Infeld theory is duality invariant [4,5] it should be possible to describe the dynamics of dyons in the same way. Note however, that the presence of a magnetic charge breaks the very basic property of the model. Namely, the field tensor $F_{\mu\nu}$ (which enters into the Lorentz equation) is no longer bounded in the vicinity of a particle. Nevertheless, as we shall show in this letter, the dyon's dynamics is well defined. We do so not only for an aesthetic purpose. It turns out that the Dirac idea of magnetically charged particles [6] (see also [7] for a review) has recently led to very remarkable results in field and string theories (see e.g. [8]).

Moreover, we present a canonical formalism for our theory. Both Hamiltonian and Lagrangian structures are constructed. The canonical structure of the Born–Infeld theory without a charged matter was already analysed by Born and Infeld [11] (see also [12]). In the presence of dyons, treated as test particles, it was discussed in [13]. The canonical structure of the theory based on the dynamical condition [1] was constructed in [9] and it is easy to show that it may be generalized to a dyon. However, in this letter we present a new structure which

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is much simpler that the one developed in [9] and, moreover, may be easily generalized to a many-particle case.

This new structure has a very interesting feature. Obviously, for a *true* dyon, the interaction is no longer based on '$j^\mu A_\mu$'. Actually, we show that there is no interaction term at all! The interaction is fully encoded into boundary conditions for the Born–Infeld field on a boundary of a punctured (dyon's position is removed) Cauchy surface. Therefore, it formally looks like a topological interaction.

It should be stressed that usually one discusses classical dyons in a different context, namely as static solutions to the (in general, non-Abelian [10]) Born–Infeld theory coupled to a Higgs field. The corresponding BPS limit [14] for such models was studied in [15]. The point-like dyon considered in this letter is simply postulated. However, there is a striking similarity between the BPS mass formula for dyons in the non-Abelian theories and a Newton-like equation (18) of this letter. We shall comment on it in the last section.

2. Dynamical condition

The Born–Infeld nonlinear electrodynamics [2] is based on the following Lagrangian (see [1] for all details of notation):

$$\mathcal{L}_{BI} := \sqrt{-\det(b\eta_{\mu\nu})} - \sqrt{-\det(b\eta_{\mu\nu} + F_{\mu\nu})} = b^2\left(1 - \sqrt{1 - 2b^{-2}S - b^{-4}P^2}\right) \tag{1}$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric with the signature $(-, +, +, +)$ and the standard Lorentz invariants S and P are defined by: $S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $P = -\frac{1}{4}F_{\mu\nu}{*F}^{\mu\nu}$ (${*F}^{\mu\nu}$ denotes the dual tensor). Adding to (1) the standard electromagnetic interaction term '$j^\mu A_\mu$' one recovers standard electromagnetic field equations:

$$\partial_\mu {*F}^{\mu\nu} = 0 \qquad \partial_\mu G^{\mu\nu} = -j^\nu \tag{2}$$

with $G^{\mu\nu} := -2\partial\mathcal{L}_{BI}/\partial F_{\mu\nu}$.

Now, let us assume that the external electric current j^μ in (2) is produced by a point-like electrically charged particle moving along the time-like trajectory ζ. Analysing the asymptotic behaviour of the fields in the vicinity of a charged particle it was shown [1] that in its rest frame the most singular part of $\boldsymbol{D}$ field behaves as

$$\boldsymbol{D}_{(-2)} = \frac{e\mathcal{A}}{4\pi}\frac{\boldsymbol{r}}{r} \tag{3}$$

where, due to the Gauss law, the monopole part of the r-independent function $\mathcal{A}$ equals 1. We use a convenient notation: $\boldsymbol{D} = r^{-2}\boldsymbol{D}_{(-2)} + r^{-1}\boldsymbol{D}_{(-1)} + \boldsymbol{D}_{(0)} + \cdots$, with r-independent $\boldsymbol{D}_{(k)}$. Moreover,

$$\boldsymbol{H} \sim r^{-1} \qquad \boldsymbol{E} \sim r^0 \qquad \boldsymbol{B} \sim r \tag{4}$$

with

$$\boldsymbol{E}_{(0)} = \frac{be}{|e|}\frac{\boldsymbol{r}}{r}. \tag{5}$$

Using these results it has been shown [1] that any regular solution of Born–Infeld field equations (2) with point-like external current satisfies:

$$\boldsymbol{E}^T_{(1)} = \frac{be}{4|e|}(3\boldsymbol{a} - r^{-2}(\boldsymbol{a}\boldsymbol{r})\boldsymbol{r}) \tag{6}$$

where $\boldsymbol{E}^T$ stands for the transversal part of $\boldsymbol{E}$ and $\boldsymbol{a}$ denotes the particle's acceleration.

Finally, the conservation of the total momentum of the composed (particle + field) system is equivalent to the following Newton-like equation:

$$ma_k = \frac{|e|b}{3}\mathcal{A}_k \tag{7}$$

where $\mathcal{A}_k$ is the dipole part of $\mathcal{A}$, i.e. $\mathrm{DP}(\mathcal{A}) =: \mathcal{A}_k x^k/r$. However, it could not be interpreted as the Newton equation because its rhs is not *a priori* given (it must be calculated from field equations). To correctly interpret (7) we have to take into account (6). Now, calculating $\boldsymbol{a}$ in terms of $\boldsymbol{E}^T_{(1)}$ and inserting into (7) we obtain the following relation between $\boldsymbol{E}^T_{(1)}$ and $\boldsymbol{D}_{(-2)}$:

$$\mathrm{DP}(4r_e^4(\boldsymbol{E}^T_{(1)})^r - \lambda_e(\boldsymbol{D}_{(-2)})^r) = 0 \tag{8}$$

where

$$r_e := \sqrt{|e|/4\pi b} \qquad \lambda_e := e^2/6\pi m. \tag{9}$$

$(\boldsymbol{F})^r$ denotes the radial component of a three-vector $\boldsymbol{F}$. The formula (8) serves as a boundary condition along ζ for the field dynamics along defined outside ζ. We call it the *dynamical condition* because it replaces particle's equations of motion. Obviously, equation (7) may be rewritten in a covariant form as a Lorentz-like equation:

$$ma^\mu = \frac{e}{r_e^2}\,\widetilde{G}^{\mu\nu}_{(-2)}u_\nu \tag{10}$$

where $\widetilde{X} = \int_{S(1)} X\,\mathrm{d}\sigma$ denotes a mean value of X on a unit sphere in the hyperplane orthogonal to u^μ centred at the particle's position.

3. Duality invariance

To show that the above theory can be generalized to also include Born–Infeld dyons let us introduce a complex notation

$$\boldsymbol{X} := \boldsymbol{D} + \mathrm{i}\boldsymbol{B} \qquad \boldsymbol{Y} := \boldsymbol{E} + \mathrm{i}\boldsymbol{H}. \tag{11}$$

The duality rotations

$$\boldsymbol{X} \to \mathrm{e}^{\mathrm{i}\alpha}\boldsymbol{X} \qquad \boldsymbol{Y} \to \mathrm{e}^{\mathrm{i}\alpha}\boldsymbol{Y} \tag{12}$$

leave Born–Infeld equations invariant. Let us consider a dyon carrying electric and magnetic charges e and g, respectively. Define the complex charge:

$$Q := e + \mathrm{i}g. \tag{13}$$

Now, instead of (3) we obviously have

$$\boldsymbol{X}_{(-2)} = \frac{Q\mathcal{A}}{4\pi}\frac{\boldsymbol{r}}{r} \tag{14}$$

and instead of relations (4) we have $\boldsymbol{X} \sim r^{-2}$ and $\boldsymbol{Y} \sim r^{-1}$. To obtain the duality invariant generalizations of (6) and (8) we proceed as follows: observe that $\bar{Q}\boldsymbol{Y}$ is duality invariant ($\bar{Q}$ stands for the complex conjugation of Q). Therefore, its real and imaginary parts are also invariant. Using this invariance let us make the duality rotation $Q' = \mathrm{e}^{\mathrm{i}\alpha}Q = e'$ (i.e. $g' = 0$) and calculate

$$\mathrm{Re}\,(\bar{Q}\boldsymbol{Y}) = e\boldsymbol{E} + g\boldsymbol{H} = e'\boldsymbol{E}'. \tag{15}$$

But in the rotated frame (i.e. $(e', 0)$) we may use results of the previous section: formula (5) implies

$$\mathrm{Re}\,(\bar{Q}\boldsymbol{Y}_{(0)}) = b|Q|\frac{\boldsymbol{r}}{r} \tag{16}$$

and (6) leads to

$$\mathrm{Re}\,(\bar{Q}\boldsymbol{Y}^T_{(1)}) = \frac{b}{4}|Q|(3\boldsymbol{a} - r^{-2}(\boldsymbol{a}\boldsymbol{r})\boldsymbol{r}). \tag{17}$$

Now, instead of (7) we have duality invariant

$$ma_k = \frac{|Q|b}{3}\mathcal{A}_k \tag{18}$$

and finally the duality invariant dynamical condition reads

$$\mathrm{DP}\{\mathrm{Re}\,[\bar{Q}(4r_0^4(\boldsymbol{Y}^T_{(1)})^r - \lambda_0(\boldsymbol{X}_{(-2)})^r)]\} = 0 \tag{19}$$

where

$$r_0^4 := r_e^4 + r_g^4 = \frac{e^2 + g^2}{(4\pi b)^2} = \frac{Q\bar{Q}}{(4\pi b)^2} \tag{20}$$

$$\lambda_0 := \lambda_e + \lambda_g = \frac{e^2 + g^2}{6\pi m} = \frac{Q\bar{Q}}{6\pi m} \tag{21}$$

are generalizations of (9). The covariant Lorentz-like equation (10) has the following duality invariant generalization:

$$ma^\mu = \frac{1}{r_0^2}(e\tilde{G}^{\mu\nu}_{(-2)} - g{*}\tilde{F}^{\mu\nu}_{(-2)})u_\nu. \tag{22}$$

4. Canonical formulation

Now we show that the duality invariant dynamical condition (19) may be derived from the mathematically well-defined variational principle. In the absence of a magnetic charge one could guess that such a principle should be based on the following Lagrangian:

$$L_{total} = L_{field} + L_{particle} + L_{int} \tag{23}$$

with L_{field} given by (1), $L_{particle} = -m\sqrt{1 - v^2}$ and $L_{int} = A_\mu j^\mu$. Varying L_{total} with respect to A_μ one obviously gets field equations (2). The variation with respect to a particle's trajectory leads to the standard Lorentz equation $ma^\mu = eF^{\mu\nu}u_\nu$. However, despite the fact that $F^{\mu\nu}$ is bounded, it is not regular at the particle's position and, therefore, the rhs of this equation is not well defined. This was already our motivation to find the mathematically well-defined dynamical condition (8) which replaces ill-defined Lorentz equations of motion. Observe that when $g \neq 0$ the situation is even worse since now both $F^{\mu\nu}$ and $G^{\mu\nu}$ behave as r^{-2}.

Let us consider a Born–Infeld dyon $(m, Q = e + \mathrm{i}g)$. The energy of the composed 'dyon + field' system in a fixed inertial laboratory frame is given by:

$$H = \sqrt{m^2 + \boldsymbol{p}^2} + V(\boldsymbol{q}) \tag{24}$$

where $\boldsymbol{p} = m\boldsymbol{v}/\sqrt{1 - \boldsymbol{v}^2}$ denotes a 'kinetic' momentum of a dyon and the function

$$V(\boldsymbol{q}) := \int_{\{\boldsymbol{q}\}} \mathrm{d}^3x\, T^{00} \tag{25}$$

defines the energy of the field configuration $(\boldsymbol{X}, \bar{\boldsymbol{X}})$ (T^{00} denotes corresponding component of an energy–momentum tensor of the Born–Infeld theory). The integral in (25) is defined on a punctured three-dimensional (constant time) space where the position of a dyon $\{\boldsymbol{q}\}$ is excluded. Moreover, in the phase space of our system, parametrized by $(\boldsymbol{q}, \boldsymbol{p})$ and by $(\boldsymbol{X}, \bar{\boldsymbol{X}})$, define the following Poisson bracket [11]:

$$\{\mathcal{F}, \mathcal{G}\} := \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \cdot \frac{\partial \mathcal{G}}{\partial \boldsymbol{p}} + \frac{1}{\mathrm{i}} \int_{\{q\}} \mathrm{d}^3 x \left[\frac{\delta \mathcal{F}}{\delta \boldsymbol{X}} \cdot \nabla \times \frac{\delta \mathcal{G}}{\delta \bar{\boldsymbol{X}}} - (\boldsymbol{X} \rightleftharpoons \bar{\boldsymbol{X}}) \right] - (\mathcal{F} \rightleftharpoons \mathcal{G}) \tag{26}$$

for any two functionals $\mathcal{F}$ and $\mathcal{G}$. With this definition we have the following commutation relations between dyon's and field variables:

$$\{q^k, p_l\} = \delta^k_l \tag{27}$$

$$\{X_k(\boldsymbol{x}), \overline{X}_l(\boldsymbol{y})\} = 2\mathrm{i}\epsilon_{klm}\partial^m \delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{28}$$

and remaining brackets vanish.

Let us treat H given by (24) as the 'dyon + field' Hamiltonian $H = H(\boldsymbol{q}, \boldsymbol{p}|\boldsymbol{X}, \bar{\boldsymbol{X}})$ and look for the corresponding Hamilton equations. In the 'field sector' everything is clear: field equations

$$\mathrm{i}\dot{\boldsymbol{X}} = \{\boldsymbol{X}, H\} = \nabla \times \boldsymbol{Y} \tag{29}$$

supplemented by the Gauss law $\nabla \cdot \boldsymbol{X} = 0$ are equivalent to the Born–Infeld field equations outside the dyon's trajectory. Observe, that $\boldsymbol{Y}$ is conjugated to $\boldsymbol{X}$ *via* $\boldsymbol{Y} = \delta H/\delta \boldsymbol{X}$. Now, in the 'dyon sector' one obviously has $\dot{\boldsymbol{q}} = \{\boldsymbol{q}, H\} = \boldsymbol{v}$. The only nontrivial thing is to evaluate

$$\dot{\boldsymbol{p}} = \{\boldsymbol{p}, H\} = -\frac{\partial}{\partial \boldsymbol{q}} V(\boldsymbol{q}). \tag{30}$$

Now, the Poincaré algebra structure implies that (30) is equivalent to

$$ma_k = \frac{|Q|b}{3}\mathcal{A}_k. \tag{31}$$

Therefore, (31) is equivalent to the dynamical condition. This way we have proved that the Hamiltonian (24) together with the Poisson bracket (26) define the consistent canonical structure of a Born–Infeld dyon.

Let us observe that there is no 'interaction term' in (24). All information about the interaction between dyon and the field is encoded in the boundary condition for the field variables which has to be satisfied near dyon's position $\boldsymbol{q}$, i.e. on the boundary of the punctured (constant time) space. From the point of view of dyon's dynamics the function (25) plays the role of a potential energy stored in the 'field sector'.

Performing the Legendre transformation in the 'dyon sector' one gets the corresponding Lagrange function

$$L(\boldsymbol{q}, \boldsymbol{v}) = -m\sqrt{1 - \boldsymbol{v}^2} - V(\boldsymbol{q}). \tag{32}$$

It is clear that Euler–Lagrange equations implied by L are equivalent to the dynamical condition for dyon's dynamics. From the point of view of dyon's dynamics the structure of (32) is evident: 'kinetic energy – potential energy'. But in the 'field sector' (32) still generates the Hamiltonian dynamics because the field generator is given by (25). Therefore, (32) is a nice example of a mixed generator called a Routhian function in analytical mechanics.

5. Concluding remarks

Finally, let us make a few remarks:

(1) Let us observe that the force in a Newton-like equation (18) does not depend on a sign of electric and magnetic charges (contrary to the standard Lorentz equation). It is a characteristic feature of the self-interaction force already present in the Lorentz–Dirac equation: $ma^\mu = eF^{\mu\nu}_{ext}u_\nu + \lambda_e(\dot{a}^\mu - a^2u^\mu)$. The external force $eF^{\mu\nu}_{ext}u_\nu$ does depend on a sign of e but a self-force proportional to λ_e does not ($\lambda_e \sim e^2$).

(2) The mass of dyon solution in the non-Abelian Yang–Mills–Higgs theory in the BPS limit is given by

$$M_{BPS} = a|Q| \tag{33}$$

where a stands for the vacuum expectation value of the Higgs field (see [7, 8, 14]). Now, observe that the lhs of Newton-like equation (7) contains purely mechanical quantities—mass m and acceleration a^k, whereas its rhs contains only electromagnetic quantities. The quantity $b|Q|/3$ looks formally like a BPS mass with $a = b/3$. With this identification (18) could be rewritten in a suggestive form:

$$ma_k = M_{BPS}\mathcal{A}_k. \tag{34}$$

Of course we do not claim that this identification has any fundamental meaning. However, our observation is supported by the fact that in string theory the b-parameter of Born–Infeld action arises as a function of a vacuum expectation value of a dilaton field.

(3) The remarkable feature of the Hamiltonian (24) and Lagrangian (32) is the absence of an interaction term. After removing a dyon's position the nontrivial topology of the space $\boldsymbol{R}^3 - \{\boldsymbol{q}\}$ requires very nontrivial boundary conditions for the field variables at the boundary $\partial(\boldsymbol{R}^3 - \{\boldsymbol{q}\})$. Therefore, in a sense, the interaction is implied by a space-time topology. Nevertheless, the above theory is not of the topological type, i.e. it is not true that its action does not depend on a space-time metric.

(4) We have eliminated a gauge potential from the variational principle and there is no need to introduce the Dirac string. Nevertheless, the standard quantization condition may be easily derived (see e.g. [7]).

(5) These results may be generalized to a many-particle case.

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